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Abstract
We analyse several integrable systems in zero-curvature form within the
framework of $SL(2, \mathbb{R})$ invariant gauge theory. In the Drinfeld–Sokolov
gauge, we derive a two-parameter family of nonlinear evolution equations
which as special cases include the Korteweg–de Vries (KdV) and Harry
Dym (HD) equations. We find residual gauge transformations which lead
to infinitesimal symmetries of this family of equations. For KdV and HD
equations we find an infinite hierarchy of such symmetry transformations,
and we investigate their relation with local conservation laws, constants of
the motion and the bi-Hamiltonian structure of the equations. Applying
successive gauge transformations of Miura type we obtain a sequence of gauge-
equivalent integrable systems, among them the modified KdV and Calogero
KdV equations.

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1. Introduction

Integrable systems in 1 + 1 dimensions have been discussed extensively so far, and various
techniques to analyse their integrability have been developed using the Lax or zero-curvature
representation, the inverse scattering method, the Hirota's bilinear method, the Painlevé
property, etc [1]. Integrable systems have also been investigated within the framework of
factorization problems on loop groups and infinite dimensional Grassmannian spaces [2].
Gauge theories play a particular role in the investigation of integrable systems. In (1 + 1) and

higher dimensions, there is a well-known conjecture by Ward [3] that any integrable systems are derived from (anti) self-dual Yang–Mills equations.

In this paper, we examine several integrable systems from the point of view of $SL(2, \mathbb{R})$ invariant gauge theory. We consider infinitesimal gauge transformations acting on $sl(2, \mathbb{R})$-valued fields which satisfy the zero-curvature equation. We analyse the transformations which leave the given gauge fixing conditions invariant (i.e. residual transformations), and thus generate infinitesimal symmetries of evolution equations. Recall that if $u(x, t)$ is a solution of the evolution equation $u_t = K[u]$, where $K[u]$ depends on x, u and a finite number of x-derivatives of u, then an infinitesimal symmetry is a transformation $u(x, t) \mapsto \widetilde{u}(x, t) = u(x, t) + \tau y(x, t)$ such that the deformed function $\widetilde{u}(x, t)$ is also a solution of the given equation to first order in τ. At least formally the infinitesimal transformations allow one to generate new solutions from the old ones. Symmetries of evolution equations have also been studied by prolonged group actions on jet spaces, as discussed by Olver [4]. In this approach, a local group of transformations acts on the space of independent and dependent variables, whereas gauge transformations considered here are globally defined groups acting on the space of dependent variables. For a particular class of evolution equations, we investigate the relationship between residual gauge transformations, conservation laws and bi-Hamiltonian structure. We show that the conservation laws and the bi-Hamiltonian structure of the Korteweg–de Vries (KdV) and Harry Dym (HD) equations are directly related to an infinite hierarchy of residual transformations, and we thus provide a new interpretation of some previously known results from a unified perspective. We also investigate the Miura transformations from the gauge theoretical point of view. Since we do not incorporate the spectral parameter, i.e. we do not consider the loop algebra $sl(2, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, the class of evolution equations obtained here is different from that considered in [5].

The outline of the paper is as follows. In section 2, we consider the Drinfeld–Sokolov gauge on the Lie algebra $sl(2, \mathbb{R})$ and we derive a two-parameter family of nonlinear evolution equations which include as special cases the well-known KdV and HD equations. We find residual gauge transformations which lead to infinitesimal symmetries of this family of equations. A different derivation of KdV as a reduction of the self-dual Yang–Mills equations on $\mathbb{R}^4$ was given in [6] (see also [7]). In section 3, we examine more closely the residual transformations for the KdV equation. We prove the existence of an infinite hierarchy of such transformations, and we show that each transformation generates a local conservation law. We further show that the residual transformations are related to an infinite hierarchy of first integrals found in [8] and the Lenard recursion operator for the bi-Hamiltonian structure of KdV. Furthermore, by applying successively the Miura-type gauge transformations to the KdV equation we obtain a sequence of gauge-equivalent integrable systems, among them are the modified KdV (mKdV) and Calogero KdV (CKdV) equations. By using residual transformations, we find nonlocal symmetries of the mKdV equation. Similar considerations are applied to the HD equation in section 4. Finally, in section 5, we note that by changing the gauge fixing condition and generalizing the gauge group to $SL(2, \mathbb{C})$ one obtains other classical integrable systems such as the sine-Gordon and nonlinear Schrödinger equation.

2. General framework

Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Consider an overdetermined linear system of equations for a wavefunction ψ,

$$\frac{\partial \psi}{\partial x_k} = A_k \psi, \qquad 1 \leqslant k \leqslant m, \tag{1}$$

where $A_k(x) \in \mathfrak{g}$ is an $n \times n$ matrix-valued gauge field and $\psi(x)$ is a vector-valued function of $x \in \mathbb{R}^m$. The consistency condition $\partial_k \partial_l \psi = \partial_l \partial_k \psi$ requires that the gauge fields satisfy the zero-curvature condition [9]

$$\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} + [A_k, A_l] = 0, \qquad 1 \leqslant k, l \leqslant m. \tag{2}$$

Equation (2) represents a hierarchy of nonlinear partial differential equations (PDEs) for components of the gauge fields. Equations which can be represented in this form are often said to be integrable, although the proof of integrability in each particular case is a nontrivial problem. The zero-curvature representation is used as a basis for the application of the geometric and algebraic techniques in the study of nonlinear PDEs, and it is also used in the classification of integrable systems by Lie algebraic methods (see, e.g. [10, 11]). The Lie algebras considered here are finite dimensional, although a number of integrable systems can be written in the zero-curvature form on infinite dimensional loop algebras. In this approach, the equation at hand is integrated by solving a Riemann–Hilbert-type factorization problem on the underlying loop group (see, e.g. [12]).

The system of equations (1) and (2) is invariant under the gauge transformation

$$\psi \mapsto g\psi, \tag{3}$$

$$A_k \mapsto \Gamma_g A_k \equiv g A_k g^{-1} + \frac{\partial g}{\partial x_k} g^{-1}, \tag{4}$$

where $g \in G$. In general, this transformation changes the particular form of the equations represented by (2). Two systems of equations are said to be gauge equivalent if their zero-curvature representations are related by a gauge transformation. For examples of gauge-equivalent systems, see [10]. Gauge transformations can be used to map solutions of one system into the other [13]. In this paper, we shall consider solutions of systems (1) and (2) which are gauge equivalent. Such solutions have the pure gauge form,

$$\psi = g\psi_0, \qquad A_k = \frac{\partial g}{\partial x_k} g^{-1}, \tag{5}$$

for some constant vector $\psi_0 \in \mathbb{C}^n$ and $g \in G$. We note that, as pointed out by Wu and Yang [14], in some cases there exist gauge fields which give rise to the same field strength but are not related by any gauge transformations.

The zero-curvature condition (2) gives rise to integrable systems of a fairly general nature. However, one is usually interested in systems involving a smaller number of functions. This is achieved by imposing gauge fixing conditions which lead to a reduced set of differential equations for independent field components. In the following, we discuss integrable systems in $(1+1)$ spacetime dimensions for $sl(2, \mathbb{R})$-valued fields. By parameterizing the gauge fields explicitly, the linear problem becomes

$$\frac{\partial \psi}{\partial x} = A_1 \psi, \qquad \frac{\partial \psi}{\partial t} = A_2 \psi, \tag{6}$$

where

$$A_1 = \begin{pmatrix} R & S \\ -T & -R \end{pmatrix}, \qquad A_2 = \begin{pmatrix} p & u \\ q & -p \end{pmatrix}. \tag{7}$$

Here, $\psi = (\psi_1 \psi_2)^t$, and the minus sign for T is introduced for convenience. The zero-curvature condition (2) yields

$$R_t - p_x = -qS - uT, \tag{8}$$

$$S_t - u_x = 2pS - 2uR, \tag{9}$$

$$T_t + q_x = -2qR - 2pT, \tag{10}$$

where the subscripts denote partial derivatives. There are three gauge freedoms for A_1 and A_2 since the six field components are related by three differential equations. First, we fix two of them by requiring

$$R = 0 \qquad \text{and} \qquad S = 1. \tag{11}$$

This is called the Drinfeld–Sokolov (DS) gauge [5]. In this gauge, the fields are given by

$$A_1^{(DS)} = \begin{pmatrix} 0 & 1 \\ -T & 0 \end{pmatrix}, \qquad A_2^{(DS)} = \begin{pmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - uT & \frac{1}{2}u_x \end{pmatrix}, \tag{12}$$

and the zero-curvature equation for $A_1^{(DS)}$ and $A_2^{(DS)}$ is equivalent to

$$T_t = 2u_x T + uT_x + \tfrac{1}{2}u_{xxx}. \tag{13}$$

We are interested in gauge transformations which preserve the gauge fixing condition (11), and thus leave equation (13) invariant. The existence of such residual transformations leads to symmetries of evolution equations and to an infinite hierarchy of conservation laws.

Define the map $K : sl(2, \mathbb{R}) \to \mathbb{R}^2$ by $K\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] = (a, b-1)$, and consider the subvariety $\mathcal{S}_K = \{A \in sl(2, \mathbb{R}) \mid K(A) = 0\}$ which consists precisely of the matrices in the DS gauge. We note that K is of maximal rank since the Jacobian of K is

$$J_K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

Fix $L \in sl(2, \mathbb{R})$ and consider the action of the one-parameter gauge group $G = \{\exp(\tau L) \mid \tau \in \mathbb{R}\}$ on $sl(2, \mathbb{R})$ by the gauge transformation (4). Since K is of maximal rank, G is a symmetry group of the set $\mathcal{S}_K \subset sl(2, A)$ if and only if each component of K satisfies (see, e.g. [4])

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} K_i(\Gamma_{\exp(\tau L)}A) = 0 \qquad \text{for all} \quad A \in \mathcal{S}_K, \quad i = 1, 2. \tag{15}$$

If L satisfies condition (15), then the gauge transformation $A_1^{(DS)} \mapsto \Gamma_g A_1^{(DS)}$ preserves the DS gauge and hence equation (13). In order to find L, let $\pi : sl(2, \mathbb{R}) \to \mathbb{R}^2$ denote the projection $\pi\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] = (a, b)$. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} K(\Gamma_{\exp(\tau L)}A) = \pi\left(\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \Gamma_{\exp(\tau L)}A\right), \tag{16}$$

where the expression in the parentheses defines the infinitesimal gauge transformation by L given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \Gamma_{\exp(\tau L)}A = [L, A] + \frac{\partial L}{\partial x}. \tag{17}$$

Hence, in view of condition (15), the DS gauge is preserved if and only if for every matrix A in the DS gauge, $A \in \mathcal{S}_K$, we have

$$\pi\left([L, A] + \frac{\partial L}{\partial x}\right) = 0. \tag{18}$$

Setting $A = A_1^{(DS)}$ and evaluating condition (18), we see that L can be parameterized by an arbitrary function $y(x, t)$ as

$$L = \begin{pmatrix} -\frac{1}{2} y_x & y \\ -\frac{1}{2} y_{xx} - yT & \frac{1}{2} y_x \end{pmatrix}. \tag{19}$$

The gauge transformation of $A_1^{(DS)}$ is now given by

$$\Gamma_{\exp(\tau L)} A_1^{(DS)} = A_1^{(DS)} + \tau \left([L, A_1^{(DS)}] + \frac{\partial L}{\partial x} \right) + o(\tau^2)$$

$$= \begin{pmatrix} 0 & 1 \\ -T & 0 \end{pmatrix} + \tau \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} y_{xxx} - 2y_x T - yT_x & 0 \end{pmatrix} + o(\tau^2), \quad (20)$$

hence the infinitesimal transformation of T is

$$\widetilde{T} = T + \tau \left(\frac{1}{2} y_{xxx} + 2y_x T + yT_x \right). \tag{21}$$

Similarly, the gauge transformation of $A_2^{(DS)}$ yields

$$\Gamma_{\exp(\tau L)} A_2(DS) = A_2^{(DS)} + \tau \left([L, A_2^{(DS)}] + \frac{\partial L}{\partial t} \right) + o(\tau^2)$$

$$= \begin{pmatrix} -\frac{1}{2} u_x & u \\ -\frac{1}{2} u_{xx} - uT & \frac{1}{2} u_x \end{pmatrix}$$

$$+ \tau \begin{pmatrix} \frac{1}{2} u_{xx} y + \frac{1}{2} u y_{xx} - \frac{1}{2} y_{xt} & y_t - y_x u + yu_x \\ \frac{1}{2} u_x y_{xx} - \frac{1}{2} u_{xx} y_x + (yu_x - y_x u)T - \frac{1}{2} y_{xxt} - (yT)_t & -\frac{1}{2} u_{xx} y - \frac{1}{2} u y_{xx} + \frac{1}{2} y_{xt} \end{pmatrix}$$

$$+ o(\tau^2). \tag{22}$$

Thus, the infinitesimal transformation of u is given by

$$\widetilde{u} = u + \tau (y_t - y_x u + yu_x). \tag{23}$$

For convenience, we denote the infinitesimal deformations by $\delta T = \frac{1}{2} y_{xxx} + 2y_x T + yT_x$ and $\delta u = y_t - y_x u + yu_x$. Since the gauge transformations of $A_1^{(DS)}$ and $A_2^{(DS)}$ satisfy the zero-curvature equation (which can be also checked directly to first order in τ using equations (20) and (22)), the functions $\widetilde{T} = T + \tau \delta T$ and $\widetilde{u} = u + \tau \delta u$ satisfy equation (13) to first order in τ. Hence, the maps $T \mapsto \widetilde{T}$ and $u \mapsto \widetilde{u}$ are infinitesimal symmetries of (13).

The remaining gauge freedom is fixed by imposing a relation between T and u. We set

$$T = \frac{u^\alpha}{s}, \qquad \alpha \in \mathbb{Z} \setminus \{0\}, \quad s > 0. \tag{24}$$

The zero-curvature condition (13) then becomes

$$u_t = \frac{\alpha + 2}{\alpha} u u_x + \frac{s}{2\alpha} u^{1-\alpha} u_{xxx}. \tag{25}$$

The motivation for considering the gauge (24) is that (25) is a two-parameter family of evolution equations which, as special cases, includes some well-known integrable systems: the KdV equation for $\alpha = 1$, $s = 2$, and HD equation for $\alpha = -2$, $s = 1$.

The infinitesimal symmetry of (25) is easily found by requiring that the function y is chosen so that $\widetilde{T}$ and $\widetilde{u}$ satisfy condition (24) to first order in τ. This holds if and only if

$$\delta T = \frac{\alpha}{s} u^{\alpha-1} \delta u. \tag{26}$$

By substituting (21) and (23) into (26), we find that y is constrained by the equation

$$y_t = \frac{\alpha + 2}{\alpha} u y_x + \frac{s}{2\alpha} u^{1-\alpha} y_{xxx}. \tag{27}$$

In this case, the infinitesimal transformation for u is given by

$$\widetilde{u} = u + \tau \left(\frac{2}{\alpha} u y_x + u_x y + \frac{s}{2\alpha} u^{1-\alpha} y_{xxx} \right). \tag{28}$$

Any smooth solution of equation (27) generates a residual gauge transformation of (25).

We noted earlier that equation (25) is integrable for $\alpha = 1, -2$. Then, equation (27) has infinitely many solutions which generate a hierarchy of conservation laws for (25), as discussed later. Furthermore, a common property shared by all equation (25) is the existence of travelling wave solutions. We conjecture that equation (25) is integrable for all $\alpha \in \mathbb{Z} \backslash \{0\}$, which is certainly true if (27) has an infinite number of solutions for all such α. For a discussion of integrability of (25) using the Painlevé property, see [15]. It is also noted that for $\alpha = 1$ the matrices $A_1^{(DS)}$ and $A_2^{(DS)}$ can be extended to matrix polynomials in the loop algebra $sl(2, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ appearing in the Riemann–Hilbert factorization for the KdV equation (see [12]). It would be a very interesting problem to determine if this construction carries over for all α and if (25) can be integrated by solving a Riemann–Hilbert factorization problem.

As stated earlier, equation (25) admit travelling wave solutions $u(x + vt)$. They are given explicitly for $\alpha = \pm 1, \pm 2$, and in implicit form for other values of α. The solutions are assumed to satisfy the boundary conditions $u, u', u'' \to 0$ as $|x + vt| \to \infty$. For $\alpha = 2k - 1, k \in \mathbb{N}$, the solution is given implicitly by

$$\sqrt{1 - bu} F\left(\frac{1}{2}, k, \frac{3}{2}; 1 - bu \right) = \frac{a}{2b^{k-1}} (x + vt + x_0), \tag{29}$$

where

$$a = \sqrt{\frac{4v}{s(\alpha + 1)}}, \qquad b = \frac{\alpha}{sv}, \tag{30}$$

and F is the hypergeometric function [16],

$$F(\alpha, \beta, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}. \tag{31}$$

Similarly, for $\alpha = 2k, k \in \mathbb{N}$, we find

$$\sqrt{1 - bu} F\left(\frac{1}{2}, \frac{1}{2} - k, \frac{3}{2}; 1 - bu \right) = \frac{a}{2b^{k-\frac{1}{2}}} (x + vt + x_0). \tag{32}$$

For $\alpha = 1$ (and $s = 2$), equation (29) yields the well-known one-soliton solution for KdV,

$$u(x, t) = 2v \operatorname{sech}^2\left(\sqrt{\frac{v}{4}} (x + vt + x_0) \right), \tag{33}$$

while for $\alpha = 2$ it gives a rational solution

$$u(x, t) = \frac{4}{a^2 (x + vt + x_0)^2 + 4b}. \tag{34}$$

The travelling wave solutions for negative integers α are given by similar expressions. For $\alpha = -(2k + 1), k \in \mathbb{N}$, we have

$$\sqrt{1 - bu} F\left(\frac{1}{2}, -k, \frac{3}{2}; 1 - bu \right) = \frac{ab^{k+1}}{2} (x + vt + x_0), \tag{35}$$

and for $\alpha = -2k$, $k \in \mathbb{N}\backslash\{1\}$, we find

$$\sqrt{1 - bu}\, F\left(\frac{1}{2}, \frac{1}{2} - k, \frac{3}{2}; 1 - bu\right) = \frac{ab^{k+\frac{1}{2}}}{2}(x + vt + x_0). \tag{36}$$

For $\alpha = -2$ (and $s = 1$), we obtain a rational solution of the HD equation,

$$u(x, t) = (9v)^{\frac{1}{3}}(x + vt + x_0)^{\frac{2}{3}}. \tag{37}$$

When $\alpha = -1$, the solution is given as a series $u(x, t) = \sum_{n=0}^{\infty} c_n(x + vt)^n$. The coefficients c_n are found recursively from

$$\sum_{k=0}^{\infty} \alpha_k \beta_{n-k} + \frac{2v}{s}(n - 2)c_{n-2} - \frac{2}{s^2}\gamma_{n-2} = 0, \qquad n \geqslant 3, \tag{38}$$

where $\alpha_n, \beta_n, \gamma_n$ are defined by

$$\alpha_n = n(n - 1)(n - 2)c_n, \qquad \beta_n = \sum_{k=0}^{n} c_{n-k}c_k, \qquad \gamma_n = \sum_{k=0}^{n} kc_{n-k}c_k. \tag{39}$$

The recurrence relation places no restrictions on c_0, c_1 and c_2. For $n \geqslant 3$ the coefficient c_n is uniquely determined from $c_0, c_1, \ldots, c_{n-1}$, hence the solution u is nontrivial provided c_0, c_1 and c_2 are all nonzero. The existence of travelling wave solutions is a consequence of the spatio-temporal translational symmetry of equation (25). Although the travelling wave solutions are not directly related to integrability of (25), they may prove helpful in investigating the system's properties.

3. Gauge transformations and the KdV sequences

In this section, we consider more closely the residual gauge transformations for the KdV equation and their relation with the Lenard recursion operator for the bi-Hamiltonian structure of KdV. We also investigate gauge transformations between the KdV and related systems, as well as their residual transformations. If $T = u/2$ ($\alpha = 1$, $s = 2$), the gauge fields in (12) become

$$A_1^{(K)} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{pmatrix}, \qquad A_2^{(K)} = \begin{pmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{pmatrix}, \tag{40}$$

hence (25) yields

$$u_t = 3uu_x + u_{xxx}. \tag{41}$$

This is one of the equivalent forms of the KdV equation. Equation (27) gives the residual gauge transformation for the KdV equation

$$y_t = 3uy_x + y_{xxx}. \tag{42}$$

This is the linear equation associated with the KdV equation which Gardner *et al* obtained from the inverse scattering method [17]. If y is a solution of (42), then $\widetilde{u} = u + \tau\delta u$ is an infinitesimal transformation of the KdV equation, where according to (28)

$$\delta u(y) = 2uy_x + u_x y + y_{xxx}. \tag{43}$$

Equation (42) has an infinite number of solutions for the parameter function y which are related to a hierarchy of conservation laws for the KdV equation. Let $y^{(1)}$ be a solution of (42) and define the function

$$G^{(1)} \equiv \delta u(y^{(1)}) = 2uy_x^{(1)} + u_x y^{(1)} + y_{xxx}^{(1)}. \tag{44}$$

Then $G^{(1)}$ satisfies the evolution equation

$$G_t^{(1)} = G_{xxx}^{(1)} + 3u_x G^{(1)} + 3u G_x^{(1)}. \tag{45}$$

This can be seen by replacing the time derivatives in $G_t^{(1)}$ by (41) and (42), and expressing $G_t^{(1)}$ only in terms of the x-derivatives of u and $y^{(1)}$. We note that equation (45) can be written as a local conservation law

$$\frac{\partial G^{(1)}}{\partial t} = \frac{\partial F^{(1)}}{\partial x} \tag{46}$$

with density $G^{(1)}$ and flux $F^{(1)} = G_{xx}^{(1)} + 3u G^{(1)}$. Now consider the function $y^{(2)}$ defined by $y_x^{(2)} = G^{(1)}$. The conservation law (46) implies that time evolution of $y^{(2)}$ is given by

$$y_t^{(2)} = \int G_t^{(1)} \, dx = 3u y_x^{(2)} + y_{xxx}^{(2)}, \tag{47}$$

which is of the same form as (42). Hence, $y^{(2)}$ also generates a residual gauge transformation for KdV. By iterating the above procedure, we obtain an infinite hierarchy of residual transformations $y^{(n)}$ defined recursively by

$$y_x^{(n+1)} = G^{(n)}, \qquad G^{(n)} = \delta u(y^{(n)}), \qquad n \in \mathbb{N}, \tag{48}$$

together with local conservation laws

$$\frac{\partial G^{(n)}}{\partial t} = \frac{\partial F^{(n)}}{\partial x}, \qquad F^{(n)} = G_{xx}^{(n)} + 3u G^{(n)}, \qquad n \in \mathbb{N}. \tag{49}$$

To illustrate the point, we start with the simplest solution $y^{(1)} = 1$. Then the first four residual transformations yield

$$y^{(1)} = 1, \tag{50}$$

$$y^{(2)} = u, \tag{51}$$

$$y^{(3)} = \tfrac{3}{2}u^2 + u_{xx}, \tag{52}$$

$$y^{(4)} = \tfrac{15}{6}u^3 + \tfrac{5}{2}u_x^2 + 5u u_{xx} + u_{xxxx}. \tag{53}$$

We remark that the functions $y^{(n)}$ are related to the isospectral problem for the Schrödinger equation, as shown in [18]. One can prove that if the eigenvalue λ in the Schrödinger equation $\Psi_{xx} + \frac{1}{6}u\Psi = \lambda\Psi$ is t-independent, then u must satisfy the evolution equation $u_t = \delta u(y^{(n)})$ for some $n \in \mathbb{N}$. In this way, starting from the isospectral problem for the Schrödinger operator, one can associate to KdV a hierarchy of integrable equations. A different approach to derive integrable hierarchies using Sato's pseudo-differential operator was given in [19].

Next we show that the functions $y^{(n)}$ are related to an infinite hierarchy of conserved quantities for the KdV equation as follows. Let $u_t = K[u]$ be an evolution equation for $u(x, t)$ and suppose that the functional

$$\mathcal{H}_n[u] = \int_{-\infty}^{\infty} H_n[u] \, dx, \qquad n \geqslant 1 \tag{54}$$

is a conserved quantity under the flow $u_t = K[u]$. In [8], Wadati showed that the variational derivative of $\mathcal{H}_n$ satisfies the integral equation

$$\int_{-\infty}^{\infty} \left(h(x) \frac{\partial}{\partial t} \left(\frac{\delta \mathcal{H}_n}{\delta u} \right) + dK(u, h) \frac{\delta \mathcal{H}_n}{\delta u} \right) dx = 0 \tag{55}$$

for every function $h(x)$ with compact support, where $dK(u, h)$ is the differential $dK(u, h) = \frac{d}{d\tau}\big|_{\tau=0} K(u + \tau h)$. If we take $K[u] = 3u u_x + u_{xxx}$, then (55) implies that $\delta \mathcal{H}_n/\delta u$ satisfies

equation (42). Thus, to each conserved quantity $\mathcal{H}_n$ of the KdV equation we can associate a residual transformation by

$$y^{(n)} = \frac{\delta \mathcal{H}_n}{\delta u}. \tag{56}$$

It is worth noting that there is a close connection between the functions $G^{(n)}$, $y^{(n)}$ and the Lenard recursion operator for the bi-Hamiltonian structure of KdV [4]. Indeed, in view of (43) we have $G^{(n)} = E y^{(n)}$, where E is the differential operator $E = D^3 + 2uD + u_x$ and D denotes the derivative with respect to x. Then (48) implies

$$G^{(n)} = R G^{(n-1)}, \tag{57}$$

where $R = ED^{-1}$ is precisely the Lenard operator for the KdV equation. We also note that

$$y^{(n)} = R^* y^{(n-1)} \tag{58}$$

where $R^* = D^{-1}E$ is the adjoint of R. This implies that the Hamiltonians $\mathcal{H}_n$ defined by (56) are related by the Lenard recursion formula

$$D\frac{\delta \mathcal{H}_n}{\delta u} = E\frac{\delta \mathcal{H}_{n-1}}{\delta u}. \tag{59}$$

Solving equation (56) for $\mathcal{H}_2$ and $\mathcal{H}_3$, we obtain

$$\mathcal{H}_2[u] = \int_{-\infty}^{\infty} \frac{1}{2}u^2 \, \mathrm{d}x, \qquad \mathcal{H}_3[u] = \int_{-\infty}^{\infty} \left(-\frac{1}{2}u_x^2 + \frac{1}{2}u^3\right) \mathrm{d}x. \tag{60}$$

The Hamiltonians $\mathcal{H}_2$ and $\mathcal{H}_3$ give rise to the well-known bi-Hamiltonian structure for KdV since

$$u_t = D\frac{\delta \mathcal{H}_3}{\delta u} = E\frac{\delta \mathcal{H}_2}{\delta u}. \tag{61}$$

All Hamiltonians $\mathcal{H}_n$ are conserved under the KdV flow, and they are all in involution with each other with respect to the Gardner–Poisson brackets for the Hamiltonian operators D and E [20]. Hence, equation (48) can be viewed as another interpretation of the bi-Hamiltonian property of the KdV equation. Applications of gauge transformations to Hamiltonian systems from a different point of view can be found in [21].

Next, we consider the Miura transformation from the gauge theoretical point of view. Miura-type transformations are known to provide links between different integrable systems, as discussed in [22]. Recall that with the choice $T = u/2$ there is no gauge freedom left for the matrices $A_1^{(K)}$ and $A_2^{(K)}$ in equation (40), and the zero-curvature condition is equivalent with the KdV equation (41). Consider the gauge transformation of $A_1^{(K)}$ and $A_2^{(K)}$ defined by the matrix

$$g = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \tag{62}$$

for some $v = v(x, t)$. The gauge fields transform according to

$$A_1^{(M)} = g A_1^{(K)} g^{-1} + g_x g^{-1} = \begin{pmatrix} v & 1 \\ -\frac{1}{2}u - v^2 - v_x & -v \end{pmatrix}, \tag{63}$$

$$A_2^{(M)} = g A_2^{(K)} g^{-1} + g_t g^{-1} = \begin{pmatrix} uv - \frac{1}{2}u_x & u \\ -\frac{1}{2}u^2 - \frac{1}{2}u_{xx} - v^2 u + vu_x - v_t & -uv + \frac{1}{2}u_x \end{pmatrix}. \tag{64}$$

Now suppose that v and u are related by the Miura transformation

$$v^2 + v_x = -\frac{1}{2}u, \tag{65}$$

then $A_1^{(M)}$ and $A_2^{(M)}$ become

$$A_1^{(M)} = \begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix}, \qquad A_2^{(M)} = \begin{pmatrix} -2v^3 + v_{xx} & -2(v^2 + v_x) \\ -v_t - 6v^2 v_x + v_{xxx} & 2v^3 - v_{xx} \end{pmatrix} \tag{66}$$

and the zero-curvature condition is equivalent with

$$\left(\frac{\partial}{\partial x} + 2v \right) (-v_t - 6v^2 v_x + v_{xxx}) = 0. \tag{67}$$

Since the zero-curvature equation is invariant under gauge transformations, we recover the well-known result which states that if v satisfies the modified KdV (mKdV) equation

$$v_t = v_{xxx} - 6v^2 v_x, \tag{68}$$

then $u = -2(v^2 + v_x)$ solves the KdV equation. Note that the reverse implication does not necessarily hold.

By using the same arguments as in section 2, one can find the residual transformations for the mKdV equation. Define $K : sl(2, \mathbb{R}) \to \mathbb{R}^2$ by $K\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] = (b - 1, c)$, and consider the subvariety

$$\mathcal{S}_K = \{ A \in sl(2, \mathbb{R}) \mid K(A) = 0 \} \tag{69}$$

which consists of matrices of the form $A_1^{(M)}$. Define the gauge group $G = \{ \exp(\tau M) \mid \tau \in \mathbb{R} \}$ for some $M \in sl(2, \mathbb{R})$. Then, G is a symmetry group of the set $\mathcal{S}_K$ if and only if the infinitesimal gauge transformation by M satisfies

$$\pi \left([M, A] + \frac{\partial M}{\partial x} \right) = 0 \qquad \text{for all} \quad A \in \mathcal{S}_K, \tag{70}$$

where the projection $\pi : sl(2, \mathbb{R}) \to \mathbb{R}^2$ is given by $\pi\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] = (b, c)$. Evaluation of the condition (70) with $A = A_1^{(M)}$ shows that M can be parameterized by a single function $z(x, t)$ as

$$M = \begin{pmatrix} zv - \frac{1}{2} z_x & z \\ c & -zv + \frac{1}{2} z_x \end{pmatrix}, \qquad \text{where} \quad c = \exp\left(-2 \int v \, dx \right). \tag{71}$$

Now, the gauge transformation of $A_1^{(M)}$ yields

$$\Gamma_{\exp(\tau M)} A_1^{(M)} = A_1^{(M)} + \tau \left([M, A_1^{(M)}] + \frac{\partial M}{\partial x} \right) + o(\tau^2) \tag{72}$$

$$= \begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix} + \tau \begin{pmatrix} -\frac{1}{2} z_{xx} + (zv)_x - c & 0 \\ 0 & \frac{1}{2} z_{xx} - (zv)_x + c \end{pmatrix} + o(\tau^2), \tag{73}$$

hence

$$\widetilde{v} = v + \tau \left(-\tfrac{1}{2} z_{xx} + (zv)_x - c \right) \tag{74}$$

is an infinitesimal transformation of the mKdV equation. This transformation is nonlocal because c is defined in terms of an indefinite integral. Among the first to consider such transformations and to point out some applications were Vinogradov and Krasilshchik [23]. Recently, nonlocal symmetries for some well-known integrable systems were found in [24].

We are left with the problem of determining the evolution equation for the function z. Let $u = -2(v^2 + v_x)$ denote the $(1, 2)$ element of the matrix $A_2^{(M)}$ in equation (66). The gauge transformation of $A_2^{(M)}$ yields the infinitesimal transformation of u,

$$\widetilde{u} = u + \tau \left(z_t - u z_x + u_x z \right). \tag{75}$$

Since $\widetilde{u}$ and $\widetilde{v}$ are also related by $\widetilde{u} = -2(\widetilde{v}^2 + \widetilde{v}_x)$ to first order in τ, this implies that z satisfies the equation

$$z_t = z_{xxx} - 6(v^2 + v_x)z_x. \tag{76}$$

We note that this is just the residual symmetry of the KdV equation for the function $u = -2(v^2 + v_x)$. Therefore, if z is a solution of equation (76), then $v \mapsto \widetilde{v} = v + \tau\delta v$ is an infinitesimal symmetry of the mKdV equation where

$$\delta v = -\frac{1}{2}z_{xx} + (zv)_x - \exp\left(-2\int v\,\mathrm{d}x\right). \tag{77}$$

The gauge transformation with g of the form (62) can be repeatedly applied to obtain other integrable systems. Consider the transformations of $A_1^{(M)}$ and $A_2^{(M)}$ defined by the matrix

$$h = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}. \tag{78}$$

We have

$$A_1^{(C)} = hA_1^{(M)}h^{-1} + h_x h^{-1} = \begin{pmatrix} v + w & 1 \\ -2vw - w^2 - w_x & -(v+w) \end{pmatrix}, \tag{79}$$

$$A_2^{(C)} = hA_2^{(M)}h^{-1} + h_t h^{-1} \tag{80}$$

$$= \begin{pmatrix} -2v^3 - 2(v^2 + v_x)w + v_{xx} & -2(v^2 + v_x) \\ 2(2v^3 - v_{xx})w + 2(v^2 + v_x)w^2 - w_t & 2v^3 + 2(v^2 + v_x)w - v_{xx} \end{pmatrix}. \tag{81}$$

If we choose w so that the $(2, 1)$ element of $A_1^{(C)}$ vanishes, i.e.

$$w_x + w^2 + 2vw = 0, \tag{82}$$

then the zero-curvature condition for $A_1^{(C)}$ and $A_2^{(C)}$ yields the Calogero KdV (CKdV) equation [25]

$$w_t = w_{xxx} - \frac{1}{2}\left(3w_x^2 w^{-1} + w^3\right)_x. \tag{83}$$

Thus, by successively applying the gauge transformations of the form (62), we obtain a sequence of integrable equations called the KdV sequence. The successive Bäcklund transformations from the KdV to CKdV equation were discussed in the bilinear formalism in [26]. The same sequence of integrable equations was obtained by Pavlov [27] by considering solutions of the overdetermined system

$$\psi_{xx} = \left(\lambda - \tfrac{1}{2}u\right)\psi, \qquad \psi_t = (4\lambda + u)\psi_x - \tfrac{1}{2}u_x\psi \tag{84}$$

whose consistency condition yields the KdV equation. He showed that by expanding the wavefunction into a power series in the spectral parameter λ,

$$\psi = \exp\left(\int\left(\sum_{n=0}^{\infty} r^{(n)}\lambda^n\right)\mathrm{d}x\right), \tag{85}$$

the coefficients $r^{(n)}(x, t)$ satisfy the same sequence of equations derived above. Thus, the successive transformations with no spectral parameter in our theory correspond to the expansion of the wavefunction into power series in the spectral parameter in conventional theories.

4. The Harry Dym equation

The Harry Dym (HD) equation [28]

$$u_t = -\tfrac{1}{4}u^3 u_{xxx} \tag{86}$$

is obtained from (25) by setting $\alpha = -2$ and $s = 1$. It was discovered by Harry Dym when trying to transfer some results about the isospectral theory to the string equation [29]. The HD equation is a completely integrable Hamiltonian system which is solvable by the inverse scattering transform [30, 31], and it possesses the bi-Hamiltonian structure and infinitely many conservation laws [32].

In this case, the fields in (12) become

$$A_1^{(HD)} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{u^2} & 0 \end{pmatrix}, \qquad A_2^{(HD)} = \begin{pmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{u} & \frac{1}{2}u_x \end{pmatrix}. \tag{87}$$

In view of equations (27) and (28), the infinitesimal symmetry transformation for the HD equation is given by

$$\widetilde{u} = u + \tau\left(-uy_x + u_x y - \tfrac{1}{4}u^3 y_{xxx}\right), \tag{88}$$

where y is constrained by the equation

$$y_t = -\tfrac{1}{4}u^3 y_{xxx}. \tag{89}$$

We show that equation (89) has an infinite number of solutions for the parameter function y. Suppose $y^{(1)}$ is a solution of (89) and define $y^{(2)}$ by

$$\left(\frac{y^{(2)}}{u}\right)_x = -uy_{xxx}^{(1)}. \tag{90}$$

We claim that $y^{(2)}$ is also a solution of equation (89). Denote for the moment $y_{xxx}^{(1)} = v$ so that $y^{(2)} = -u\int uv\,dx$. Then a simple computation shows that

$$y_t^{(2)} = \frac{1}{4}u^3 u_{xxx}\int uv\,dx + \frac{1}{4}u\left(\int u_{xxx}u^3 v\,dx + \int u(u^3 v)_{xxx}\,dx\right) \tag{91}$$

where we used the assumption that $y^{(1)}$ satisfies (89), and we replaced the time derivative of u by equation (86). After a few partial integrations of the last term, we obtain

$$\int u(u^3 v)_{xxx}\,dx = u(u^3 v)_{xx} - u_x(u^3 v)_x + u_{xx}u^3 v - \int u_{xxx}u^3 v\,dx, \tag{92}$$

which finally yields

$$y_t^{(2)} = \frac{1}{4}u^3\left(u_{xxx}\int uv\,dx + 3u_x^2 v + 5uu_x v_x + 4uu_{xx}v + u^2 v_{xx}\right). \tag{93}$$

It is straightforward to verify that the expression in the parentheses equals $-y_{xxx}^{(2)}$, hence $y^{(2)}$ satisfies equation (89). Therefore, (90) provides a recursion formula for $y^{(n)}$ which generates an infinite hierarchy of residual gauge transformations for the HD equation. Starting with $y^{(1)} = 1$ the first four functions are given by

$$y^{(1)} = 1, \tag{94}$$

$$y^{(2)} = u, \tag{95}$$

$$y^{(3)} = \tfrac{1}{2}uu_x^2 - u^2 u_{xx}, \tag{96}$$

$$y^{(4)} = \tfrac{3}{8}uu_x^4 - \tfrac{3}{2}u^2 u_x^2 u_{xx} + \tfrac{3}{2}u^3 u_{xx}^2 + 2u^3 u_x u_{xxx} + u^4 u_{xxxx}. \tag{97}$$

As in the case of the KdV equation the conserved quantities of the HD equation are related to the residual gauge transformations $y^{(n)}$. Let

$$\mathcal{G}_n[u] = \int_{-\infty}^{\infty} G_n[u] \, dx, \qquad n \in \mathbb{N} \tag{98}$$

be a conserved quantity of the HD equation (86). Substituting $K[u] = -\frac{1}{4}u^3 u_{xxx}$ into equation (55) one can show that $\delta\mathcal{G}_n/\delta u$ satisfies

$$\left(\frac{\delta\mathcal{G}_n}{\delta u}\right)_t = \frac{3}{4}u^2 u_{xxx}\frac{\delta\mathcal{G}_n}{\delta u} - \frac{1}{4}\left(u^3\frac{\delta\mathcal{G}_n}{\delta u}\right)_{xxx}. \tag{99}$$

We emphasize that, unlike in the KdV case, the above equation is not the same as the residual transformation (89) for $y^{(n)}$. However, if we set

$$y^{(n)} = u^3\frac{\delta\mathcal{G}_n}{\delta u}, \tag{100}$$

then it is easily seen that $y^{(n)}$ satisfies (89) if and only if $\delta\mathcal{G}_n/\delta u$ satisfies (99). Therefore, the conserved quantities and residual gauge transformations of the HD equation are related by equation (100). The conserved quantities corresponding to the first four residual transformations $y^{(n)}$ found earlier are given by

$$\mathcal{G}_1[u] = \int_{-\infty}^{\infty} \left(-\frac{1}{2}u^{-2}\right) dx, \tag{101}$$

$$\mathcal{G}_2[u] = \int_{-\infty}^{\infty} (-u^{-1}) \, dx, \tag{102}$$

$$\mathcal{G}_3[u] = \int_{-\infty}^{\infty} \frac{1}{2}u^{-1}u_x^2 \, dx, \tag{103}$$

$$\mathcal{G}_4[u] = \int_{-\infty}^{\infty} \left(\frac{1}{8}u^{-1}u_x^4 + \frac{1}{2}uu_{xx}^2\right) dx. \tag{104}$$

5. The sine-Gordon and nonlinear Schrödinger equations

So far we have discussed integrable systems on the Lie algebra $sl(2, \mathbb{R})$ which arise by fixing the DS gauge (11). In this section, we consider briefly a few generalizations obtained by changing the gauge fixing condition and generalizing the gauge group. More precisely, we derive the sine-Gordon (SG) and focusing nonlinear Schrödinger (NLS) equations. The NLS equation was derived by symmetry reduction from the self-dual Yang–Mills equations in [6] (see also [7]). We remark that the residual transformations approach presented in the previous sections can be extended to study infinitesimal symmetries of the SG and NLS equations as well.

Consider the gauge fields (7) with $R = \alpha, \alpha \in \mathbb{R}\backslash\{0\}$ and $S = T = -\frac{1}{2}w_x$ for some $w = w(x, t)$. In order to fix the remaining gauge freedom suppose $q = u$. Then the zero-curvature equations (8)–(10) yield

$$p_x = -qw_x, \tag{105}$$

$$\tfrac{1}{2}w_{xt} + q_x = pw_x + 2\alpha q, \tag{106}$$

$$-\tfrac{1}{2}w_{xt} + q_x = pw_x - 2\alpha q. \tag{107}$$

Adding and subtracting equations (106) and (107), we obtain

$$q_x = pw_x, \tag{108}$$

$$w_{xt} = 4\alpha q. \tag{109}$$

We note that equations (105) and (108) admit solutions $p = \cos(w)$ and $q = \sin(w)$. With this choice of p and q, the gauge fields become

$$A_1^{(SG)} = \begin{pmatrix} \alpha & -\frac{1}{2}w_x \\ \frac{1}{2}w_x & -\alpha \end{pmatrix}, \qquad A_2^{(SG)} = \begin{pmatrix} \cos(w) & \sin(w) \\ \sin(w) & -\cos(w) \end{pmatrix}. \tag{110}$$

The zero-curvature condition is then equivalent with (109), which is the sine-Gordon equation

$$w_{xt} = 4\alpha \sin(w). \tag{111}$$

Similarly, we derive the focusing NLS equation by considering the gauge group $SL(2, \mathbb{C})$ and introducing the gauge fixing conditions $R = 0$, $S = \overline{T} = w$ and $q = -\overline{u}$. In this case, the zero-curvature equations (8)–(10) yield

$$-qw + \overline{q}\,\overline{w} = -p_x, \tag{112}$$

$$w_t + \overline{q}_x = 2pw, \tag{113}$$

$$\overline{w}_t + q_x = -2p\overline{w}. \tag{114}$$

Equation (112) implies that p_x is purely imaginary, hence we may impose further reduction of the gauge freedom by requiring $p = -\mathrm{i}|w|^2$. Then, equations (113) and (114) are complex conjugate of each other. Furthermore, if we set $q = -\mathrm{i}\overline{w}_x$, then relation (112) holds identically. In this case, the gauge fields are

$$A_1^{(NLS)} = \begin{pmatrix} 0 & w \\ -\overline{w} & 0 \end{pmatrix}, \qquad A_2^{(NLS)} = \begin{pmatrix} -\mathrm{i}|w|^2 & -\mathrm{i}w_x \\ -\mathrm{i}\overline{w}_x & \mathrm{i}|w|^2 \end{pmatrix}, \tag{115}$$

and (114) yields the focusing NLS equation

$$\mathrm{i}w_t - w_{xx} - 2|w|^2 w = 0. \tag{116}$$

The sine-Gordon and the focusing NLS equation can also be derived within the framework of Birkhoff factorization of loop groups. For more details, see, e.g. [13, 33].

6. Conclusion

In this paper, we have discussed integrable systems within the framework of gauge theory. The gauge theoretical structure of the Zakharov–Shabat formalism has been fully discussed. Starting with the DS gauge, we have fixed the residual gauge freedom by relation (24) and obtained a two-parameter family of nonlinear equation (25). It was shown explicitly that for $\alpha = 1$ (KdV) and $\alpha = -2$ (HD) there exist infinite hierarchies of residual gauge transformations. The transformations can be calculated recursively and they lead to an infinite number of conserved quantities. We have also shown that all equation (25) admit travelling wave solutions which are given implicitly in terms of the hypergeometric function, except for $\alpha = \pm 1, \pm 2$ when the solutions are given in explicit form. However, it is an open problem to determine if equation (25) is integrable for all $\alpha \in \mathbb{Z}\backslash\{0\}$. The Miura transformations which lead to the KdV sequence (KdV $\to$ mKdV $\to$ CKdV, etc) have been formulated in terms of gauge transformations acting on the gauge fields defined on the Lie algebra $sl(2, \mathbb{R})$. It was shown that other types of $(1 + 1)$ integrable systems can also be formulated within the same framework by changing the initial gauge fixing condition and generalizing the gauge group.

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